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Symmetric monopoles and finite-gap Lamé potentials

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Abstract. For each positive integer g, we construct a one-parameter family of spectral curves for D_2 symmetric charge $2g+1$ SU(2) BPS monopoles. Each spectral curve is reducible, and is the union of a line with g elliptic curves. We show that such a monopole is related to a g-gap Lamé potential. Other symmetric monopoles, related to elliptic curves, are also shown to have a similar correspondence. A suggestion is made on how this observation may be of use in the construction of new spectral curves.

1. Introduction

The subject of this paper is static SU(2) BPS monopoles, which are topological soliton solutions of a Yang–Mills–Higgs gauge theory in three-space dimensions. The Bogomolny equation for static monopoles is a reduction of the self-dual Yang–Mills equation, and hence a variety of twistor approaches may be taken. Rather than constructing the monopole itself, we compute the twistor data to which it is equivalent, namely its spectral curve and Nahm data. For each odd integer $n > 1$, we exhibit a spectral curve which is a one-parameter deformation of the known spectral curve of an axisymmetric n-monopole. Physically, this spectral curve describes n monopoles equally spaced along a line through the origin. The parameter in the spectral curve is related to the distance between neighbouring monopoles. This configuration has dihedral D_2 symmetry. The spectral curve is reducible, and is the union of a line with g elliptic curves, where $n = 2g+1$.

We relate such a monopole, via its Nahm data and an observation due to Ward [14], to a Lamé equation. We find that the potential is a g-gap potential where again $n = 2g+1$. Such potentials are well studied in connection with soliton solutions of the periodic KdV equation [12].

Finally, we show that some recently discovered symmetric monopoles [9], related to elliptic curves, are associated with 1-gap Lamé potentials. This leads to a suggestion on how this observation may be of use in the construction of new spectral curves.

2. Spectral curves and Nahm data

A given multi-monopole, being a topological soliton, has an associated integer winding number n, which may be thought of as giving the number of monopoles. It also equals the total magnetic charge of the monopole, in units of 4π. We refer to a monopole of charge n as an n-monopole. In this paper we are mainly concerned with two twistor approaches to monopoles, which we review in the following.

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Hitchin has shown [5, 6] that monopoles correspond to certain algebraic curves, called spectral curves, in the holomorphic tangent bundle to the Riemann sphere $\mathrm{T}\mathbb{CP}^1$. Let ζ be the standard inhomogeneous coordinate on the base space and η the fibre coordinate, then an n-monopole corresponds to a curve of the form

$$\eta^n + \eta^{n-1}a_1(\zeta) + \cdots + \eta^r a_{n-r}(\zeta) + \cdots + \eta a_{n-1}(\zeta) + a_n(\zeta) = 0 \tag{2.1}$$

where, for $1 \leqslant r \leqslant n$, $a_r(\zeta)$ is a polynomial in ζ of maximum degree $2r$ which satisfies the reality condition

$$a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r\left(-\frac{1}{\bar{\zeta}}\right)}. \tag{2.2}$$

In addition the algebraic curve must satisfy a difficult non-singularity constraint [5].

For a single monopole ($n = 1$) the non-singularity constraint is automatically satisfied, and the spectral curve is given by

$$\eta - (x_1 + \mathrm{i}x_2) + 2x_3\zeta + (x_1 - \mathrm{i}x_2)\zeta^2 = 0 \tag{2.3}$$

where (x_1, x_2, x_3) is the monopole's position in $\mathbb{R}^3$. We shall follow the notation of [4] and later refer to such a spectral curve as a star.

Let $n = 2g+1$, where g is a positive integer, then there is an axisymmetric n-monopole with spectral curve [5]

$$\eta \prod_{l=1}^{g} \{\eta^2 + l^2\pi^2\zeta^2\} = 0. \tag{2.4}$$

One may think of this spectral curve as describing n monopoles, which are all positioned at the origin (there are no spherically symmetric monopoles for $n > 1$). In this paper we shall present a one-parameter deformation of the spectral curve (2.4), which corresponds to pulling apart the individual $2g+1$ monopoles so that the axial symmetry is broken to dihedral symmetry D_2. This is achieved with the aid of the Atiyah–Drinfeld–Hitchin–Manin–Nahm (ADHMN) construction, which we now briefly describe.

The ADHMN construction [11, 6] is an equivalence between an n-monopole and Nahm data (T_1, T_2, T_3), which are three $n \times n$ matrices which depend on a real parameter $s \in [0, 2]$ and satisfy the following:

(i) Nahm's equation

$$\frac{\mathrm{d}T_i}{\mathrm{d}s} = \frac{1}{2}\epsilon_{ijk}[T_j, T_k]; \tag{2.5}$$

(ii) $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0$ and $s = 2$;

(iii) the matrix residues of (T_1, T_2, T_3) at each pole form the irreducible n-dimensional representation of SU(2);

(iv) $T_i(s) = -T_i^\dagger(s)$;

(v) $T_i(s) = T_i^t(2 - s)$.

Equation (2.5) is equivalent to a Lax pair and hence there is an associated algebraic curve, which is in fact the spectral curve. Explicitly, the spectral curve may be read off from the Nahm data as the equation

$$\det(\eta + (T_1 + \mathrm{i}T_2) - 2\mathrm{i}T_3\zeta + (T_1 - \mathrm{i}T_2)\zeta^2) = 0. \tag{2.6}$$

We shall now present the Nahm data for our D_2 symmetric monopoles, and hence compute the spectral curves.

As above, let g be an integer such that $n = 2g + 1$. Let ρ_1, ρ_2, ρ_3 be three anti-Hermitian $n \times n$ matrices of the standard spin g irreducible representation of su(2) satisfying

$[\rho_1, \rho_2] = 2\rho_3$ etc. Then, motivated by Dancer's work on SU(3) monopoles [2], we take our Nahm data to be given by

$$T_1 = -\frac{K\,\mathrm{dn}(Ks, k)}{2\mathrm{sn}(Ks, k)}\rho_1 \qquad T_2 = -\frac{K}{2\mathrm{sn}(Ks, k)}\rho_2 \qquad T_3 = -\frac{K\,\mathrm{cn}(Ks, k)}{2\mathrm{sn}(Ks, k)}\rho_3 \tag{2.7}$$

where $\mathrm{sn}(u, k)$ etc denote the Jacobi elliptic functions with modulus k, and K is the complete elliptic integral of the first kind with modulus k. It is a simple matter to check that for $0 \leqslant k < 1$ the properties (i) to (iv), required by the Nahm data, are satisfied. In the basis we have chosen, the last requirement (v) is not explicitly satisfied, but we can appeal to a general argument (see [4]) that a basis exists in which (v) is satisfied.

Using (2.6) we compute the associated spectral curves to be

$$\eta \prod_{l=1}^{g} \{\eta^2 + l^2 K^2 (4\zeta^2 - k^2(\zeta^2 + 1)^2)\} = 0. \tag{2.8}$$

Now we have the spectral curve, we can examine the symmetry of the corresponding n-monopole. Let R_i, $i = 1, 2, 3$, denote the generator of rotations by π around the x_i-axis. Then R_3 acts on $\mathrm{T}\mathbb{CP}^1$ as

$$R_3(\eta, \zeta) = (-\eta, -\zeta). \tag{2.9}$$

Under this transformation the spectral curve (2.8) is clearly the same curve, so it has cyclic C_2 symmetry around the x_3-axis. The rotation R_1 acts on $\mathrm{T}\mathbb{CP}^1$ as

$$R_1(\eta, \zeta) = (-\eta/\zeta^2, 1/\zeta) \tag{2.10}$$

and again it can be seen that the curve (2.8) is the same curve under this transformation. This R_1 symmetry extends the cyclic symmetry C_2 to dihedral symmetry D_2. Hence we have demonstrated that, for each g, the curve (2.8) is the spectral curve of a charge $2g + 1$ monopole with D_2 symmetry. Note that we also have an additional reflection symmetry $J : x_2 \to -x_2$, which acts on $\mathrm{T}\mathbb{CP}^1$ as

$$J(\eta, \zeta) = (\bar{\eta}, \bar{\zeta}) \tag{2.11}$$

as a consequence of the fact that all coefficients in the spectral curve are real.

Let us now examine two special cases for this spectral curve. The first is when $k = 0$, so that $K = \pi/2$, and the curve becomes that of the axisymmetric n-monopole given by (2.4), with x_3 the axis of symmetry. For this value of k each of the g elliptic curves in (2.8) becomes a rational curve, so that the spectral curve is the union of n sections of the bundle $\mathrm{T}\mathbb{CP}^1 \to \mathbb{CP}^1$. The second special case is the limit in which $k \to 1$. In this limit we have that $K \to \infty$, so that the spectral curve (2.8) becomes asymptotic to the product of stars

$$\prod_{l=-g}^{g} \{\eta + lK(\zeta^2 - 1)\} = 0. \tag{2.12}$$

This describes $2g + 1$ well separated monopoles along the x_1-axis. One monopole is at the origin and the remaining $2g$ monopoles are equally spaced along the positive and negative x_1-axis, with the distance between any two neighbouring monopoles being equal to K. Given these two limiting cases it is natural to think of the parameter k in the spectral curve as determining the distance between neighbouring monopoles.

Note that a similar family of spectral curves can be calculated for D_2 symmetric monopoles of even charge by taking a half odd integer spin representation of su(2). For the 2-monopole case the corresponding results were obtained some time ago [1] and the form of the Nahm data is very similar to that which is used here (2.7) to construct n-monopoles†.

† I thank one of the referees for drawing this to my attention.

In the next section we describe how our D_2 symmetric monopoles are related to finite-gap Lamé potentials.

3. The Lamé equation

The Lamé equation [3, 12] is the simplest example of a finite-gap Hill's equation. In Jacobi form it is written as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2} - g(g+1)k^2\mathrm{sn}^2(u,k)\psi = -E\psi \tag{3.13}$$

where g and E are constants and we use the same notation for Jacobi elliptic functions and integrals as in the previous section.

If g is a positive integer then the operator appearing in (3.13) has a finite number of gaps in its spectrum, in fact g of them, as explained below. The elliptic potential appearing in the above operator is periodic with period $2K$. The eigenfunction ψ is a Bloch eigenfunction if it is an eigenvector of the translation operator $u \to u + 2K$, i.e.

$$\psi(u+2K) = \mathrm{e}^{2\mathrm{i}Kp}\psi(u) \tag{3.14}$$

where p, which is a function of E, is the quasi-momentum. Now the Bloch eigenfunction only belongs to the Bloch spectrum of the operator in (3.13) if the quasi-momentum is real. Hence there are forbidden bands for the allowed values of E, corresponding to gaps in the spectrum of the operator in the Lamé equation. The result of interest is that if g is a positive integer then the number of gaps in the spectrum is finite, and is in fact equal to g. The elliptic potential in (3.13) is then called a g-gap potential and the Bloch eigenfunction is a mereomorphic function defined on a Riemann surface of genus g. These g-gap potentials arise in the solution of the periodic KdV equation [12], and are associated with the g-soliton solutions, which may be constructed using algebraic–geometric methods.

The connection between monopoles and the Lamé equation is provided by the Nahm data. Ward has observed [14] that the Lamé equation can be written in terms of the composition of two first-order matrix operators, the coefficients of which satisfy Nahm's equation (2.5). Hence to each monopole, we can associate its Nahm data, construct the first-order matrix operators, and form their composition to arrive at a Lamé equation. In this way we shall see how the D_2 symmetric monopoles of the previous section are related to g-gap potentials. Note that Ward considered an explicit example of the construction of a Lamé equation from a particular solution of Nahm's equation. However, in his example the solution of Nahm's equation did not satisfy the extra conditions ((ii) to (v) of the previous section) required of the Nahm data. In particular his chosen solution of Nahm's equation is regular for all $s \in [0, 2]$, and so does not correspond to a monopole.

From the Nahm data (T_1, T_2, T_3) construct the following linear $2n \times 2n$ matrix operator

$$\Delta = \mathbb{1}_{2n}\frac{\mathrm{d}}{\mathrm{d}s} + \mathrm{i}T_j \otimes \sigma_j \tag{3.15}$$

where $\mathbb{1}_n$ denotes the $n \times n$ identity matrix and σ_j are the Pauli matrices. One may consider (3.15) as a quaternionic operator, then an important cornerstone of the ADHM construction [11, 6] is that the composition of this operator with its conjugate is real, i.e.

$$\Delta^\star\Delta = \left(\mathbb{1}_n\frac{\mathrm{d}^2}{\mathrm{d}s^2} + T_jT_j\right) \otimes \mathbb{1}_2 \tag{3.16}$$

so that the imaginary quaternionic part is zero. This is a result of the reality properties of the Nahm data and the fact that Nahm's equation (2.5) is satisfied. Hence given Nahm data it is natural to associate it with the linear matrix differential equation

$$\mathbb{1}_n \frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} + T_j T_j \psi = 0 \tag{3.17}$$

for the n-vector ψ. This equation may be viewed as n copies of the Lamé equation.

Let us now see how all this works for the Nahm data (2.7) of a charge $2g+1$ monopole with D_2 symmetry. Using standard identities between Jacobi elliptic functions it is a simple task to show that in this case

$$T_j T_j = \frac{K^2 \mathrm{ns}^2(Ks, k)}{4}(\rho_1^2 + \rho_2^2 + \rho_3^2) - \frac{K^2}{4}(k^2\rho_1^2 + \rho_3^2). \tag{3.18}$$

Now $\rho_1^2+\rho_2^2+\rho_3^2$ is the quadratic Casimir for the spin g representation of su(2) so it is equal to $-4g(g+1)\mathbb{1}_n$. Furthermore, $\mathrm{ns}(u+\mathrm{i}K', k) = k\,\mathrm{sn}(u, k)$, where K' is the complete elliptic integral of the first kind with argument given by the complementary modulus $k' = \sqrt{1-k^2}$. Using these two results, and defining the variable $u = Ks - \mathrm{i}K'$, we have that

$$T_j T_j = -K^2 k^2 g(g+1)\mathrm{sn}^2(u, k)\mathbb{1}_n - \tfrac{1}{4}K^2(k^2\rho_1^2 + \rho_3^2). \tag{3.19}$$

Hence equation (3.17) becomes

$$\mathbb{1}_n \left(\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2} - g(g+1)k^2\mathrm{sn}^2(u, k)\psi \right) = -M\psi \tag{3.20}$$

where M is the constant matrix $-4(k^2\rho_1^2 + \rho_3^2)$. This equation is n copies of the Lamé equation with a g-gap elliptic potential.

In summary, we have constructed, for each positive integer g, a family of spectral curves for a D_2 symmetric charge $2g+1$ monopole, each of which is reducible and is the union of a line with g elliptic curves. We have then shown how each of these monopoles is related to a g-gap Lamé potential.

We now consider another example, for which it will be more convenient to write the Lamé equation in its Weierstrass form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2} - g(g+1)\wp(u)\psi = -E\psi \tag{3.21}$$

where $\wp(u)$ is the Weierstrass elliptic function satisfying

$$\wp'^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \tag{3.22}$$

with $'$ denoting differentiation with respect to the argument.

The family of monopoles we now consider were recently constructed in [9] and consist of a charge 3-monopole which is invariant under a combined 90° rotation and inversion. We computed a one-parameter family of such monopoles, which includes the axisymmetric 3-monopole and the tetrahedral 3-monopole [4, 8]. Explicitly, the family of spectral curves is given by

$$\eta^3 - 6(a^2-4)^{1/3}\kappa^2\eta\zeta^2 + 2\mathrm{i}\kappa^3 a(\zeta^5 - \zeta) = 0 \tag{3.23}$$

where $a \in \mathbb{R}$ is the parameter, and κ is a known function of a. Special cases include the axisymetric 3-monopole ($a = 0$, $\kappa = \pi/(2^{5/6}\sqrt{3})$) and the tetrahedral 3-monopole ($a = \pm 2$, $\kappa = \Gamma(1/6)\Gamma(1/3)/(4\sqrt{3\pi})$).

Now if this monopole is related to a g-gap potential, it would be useful to predict the number of gaps directly from the spectral curve. In the previous example we saw that the number of gaps g was equal to the number of elliptic curves whose union (together with a line) gave the spectral curve. The generic n-monopole spectral curve is irreducible and has genus $(n-1)^2$ [6], so the 3-monopole spectral curve (3.23) has genus 4. However, it is a particularly symmetric curve, since the 3-monopole has the symmetry group G generated by a reflection symmetry in the $x_1 = x_2$ plane, $\sigma : (x_1, x_2, x_3) \to (x_2, x_1, x_3)$, and by a combined 90° rotation and inversion $\mathcal{T} : (x_1, x_2, x_3) \to (x_2, -x_1, -x_3)$. The twisted inversion operator $\mathcal{T}$ and the reflection operator σ act on $\mathrm{T}\mathbb{CP}^1$ as

$$\mathcal{T}(\eta, \zeta) = (-\mathrm{i}\bar{\eta}/\bar{\zeta}^2, \mathrm{i}/\bar{\zeta}) \tag{3.24}$$

$$\sigma(\eta, \zeta) = (\mathrm{i}\bar{\eta}, \mathrm{i}\bar{\zeta}). \tag{3.25}$$

We now show, by computing Euler characteristics, that the quotient of the spectral curve (3.23) by G is an elliptic curve. By the Riemann–Hurwitz formula the Euler characteristic χ of the spectral curve is related to the Euler characteristic $\widetilde{\chi}$ of the quotient curve by

$$\chi = |G|\widetilde{\chi} - b \tag{3.26}$$

where b is the total branching. The symmetry group G acts freely except when $\eta = 0$, when there are fixed points at $\zeta = 0$ and $\zeta = \infty$. Each of these fixed points has a stabilizer of order 4, hence the total branching $b = 2(4-1) = 6$. Since the spectral curve has genus 4 we have that $\chi = -6$ giving

$$-6 = |G|\widetilde{\chi} - 6 \tag{3.27}$$

so that $\widetilde{\chi} = 0$ and the quotient curve is elliptic.

Given the previous example, where the spectral curve consisted of g elliptic curves and the Lamé potential had g gaps, the fact that the quotient curve is elliptic suggests that perhaps this family of monopoles is related to a 1-gap potential. We now show that this is indeed the case.

The Nahm data for this family of monopoles has the form [9]

$$T_1 = \mathrm{i}\beta\sqrt{2}\begin{bmatrix} 0 & \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ \mathrm{e}^{\mathrm{i}\theta} & 0 & \mathrm{e}^{\mathrm{i}\theta} \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} & 0 \end{bmatrix} \qquad T_2 = \beta\sqrt{2}\begin{bmatrix} 0 & \mathrm{e}^{\mathrm{i}\theta} & 0 \\ -\mathrm{e}^{-\mathrm{i}\theta} & 0 & \mathrm{e}^{-\mathrm{i}\theta} \\ 0 & -\mathrm{e}^{\mathrm{i}\theta} & 0 \end{bmatrix}$$

$$T_3 = 2\alpha\begin{bmatrix} \mathrm{i}\cos\phi & 0 & -\sin\phi \\ 0 & 0 & 0 \\ \sin\phi & 0 & -\mathrm{i}\cos\phi \end{bmatrix} \tag{3.28}$$

where $\alpha, \beta, \theta, \phi$ are some known functions of s. To construct the associated Lamé equation we compute that

$$T_jT_j = -4\begin{bmatrix} \alpha^2+\beta^2 & 0 & 0 \\ 0 & 2\beta^2 & 0 \\ 0 & 0 & \alpha^2+\beta^2 \end{bmatrix} = -4(\alpha^2+\beta^2)\mathbb{1}_3 + 4(\alpha^2-\beta^2)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.29}$$

In the above the decomposition, into a piece proportional to the identity and a remainder, reflects the fact that we require the same Lamé potential for each copy of the Lamé equation. Now it is clear that to obtain a Lamé equation the combination $\alpha^2 + \beta^2$ must be (up to a constant) given by an elliptic function, and the remainder coefficient $\alpha^2 - \beta^2$ must be constant. The explicit solution (from [9]) is

$$\alpha(s) = \frac{\kappa}{2}\sqrt{\wp(\kappa s) + (a^2-4)^{1/3}} \tag{3.30}$$

$$\beta(s) = \frac{\kappa}{2}\sqrt{\wp(\kappa s) - \tfrac{1}{2}(a^2-4)^{1/3}} \tag{3.31}$$

where $\wp$ is the Weierstrass elliptic function satisfying

$$\wp'^2 = 4\wp^3 - 3(a^2-4)^{2/3}\wp - 4. \tag{3.32}$$

Thus

$$\alpha^2 + \beta^2 = \frac{\kappa^2}{2}\left(\wp + \frac{(a^2-4)^{1/3}}{4}\right) \tag{3.33}$$

$$\alpha^2 - \beta^2 = \frac{3\kappa^2}{8}(a^2-4)^{1/3} \tag{3.34}$$

and we have the required behaviour. Note that in solving Nahm's equation to obtain the solutions (3.30) it is a non-trivial exercise to find the appropriate combination of functions which obey a Weierstrass equation. However, using the relation to a Lamé potential we have seen that the appropriate combination of $\alpha^2 + \beta^2$ emerges naturally. Finally, defining the variable $u = \kappa s$ equation (3.17) becomes the 1-gap Lamé equation

$$\mathbb{1}_3\left(\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2} - 2\wp(u)\psi\right) = -M\psi \tag{3.35}$$

where $M = \frac{1}{2}(a^2-4)^{1/3}\mathrm{diag}(-1, 2, -1)$ is a constant matrix.

This example suggests that constructing the corresponding Lamé equation for a monopole may be of use in solving Nahm's equation, since it could naturally lead to a simplifying choice of variable combinations. As an example we consider the Nahm data for a charge 3-monopole with cyclic C_3 symmetry. It is known that the Nahm data has the form

$$T_1 + \mathrm{i}T_2 = \begin{bmatrix} 0 & 0 & \alpha_2 \\ \alpha_3 & 0 & 0 \\ 0 & \alpha_1 & 0 \end{bmatrix} \qquad T_3 = \mathrm{i}\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \tag{3.36}$$

and Nahm's equation is equivalent to the 3-particle Toda chain [7]. However, despite the fact that these equations are a much-studied integrable system, it has not as yet proved tractable in this case to find the explicit solution of Nahm's equation which satisfies all the required boundary conditions.

With a view to constructing the associated Lamé equation we compute that

$$T_jT_j = -\frac{1}{2}\begin{bmatrix} 2d_1^2 + \alpha_2^2 + \alpha_3^2 & 0 & 0 \\ 0 & 2d_2^2 + \alpha_1^2 + \alpha_3^2 & 0 \\ 0 & 0 & 2d_3^2 + \alpha_1^2 + \alpha_2^2 \end{bmatrix}. \tag{3.37}$$

We can make this proportional to the identity matrix by the obvious choice $2d_i^2 = \alpha_i^2$, $i = 1, 2, 3$. This reduction simplifies the equations sufficiently to be integrated explicitly, in terms of an elliptic function, and results in the spectral curve

$$\eta^3 + \mathrm{i}\Gamma(1/6)^3\Gamma(1/3)^3/(\pi^{3/2}3^3 8\sqrt{2})(\zeta^6 + \mathrm{i}5\sqrt{2}\zeta^3 + 1) = 0. \tag{3.38}$$

It can be shown that this is the spectral curve of the tetrahedral 3-monopole [4], after a rotation so that the tetrahedral monopole has a cyclic C_3 symmetry around the x_3-axis. So, by considering the associated Lamé equation we have been lead to a simplified case. Perhaps other cases may be constructed in a similar fashion.

4. Conclusion

A new family of spectral curves, for odd charge monopoles with dihedral symmetry, has been computed and a relation with finite-gap Lamé potentials investigated. It would be interesting to see if the connection between monopoles and g-gap potentials could be made more formal. In particular g-gap potentials can be associated, via the Novikov equation [13, 12], to algebraic curves. Perhaps a link between these curves and the spectral curves of the monopole could be found.

Note that in general the problem of determining which monopoles correspond to finite-gap potentials, and conversely which finite-gap potentials can arise, is an open problem.

Another aspect to investigate is the role that could be played by the Lamé polynomials [3] within the ADHMN construction.

Finally, the simplest example studied in this paper, that of three monopoles with D_2 symmetry, can be shown to be a totally geodesic submanifold of the 3-monopole moduli space. This work will be presented elsewhere [10].

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